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Integrable quasiclassical deformations of algebraic curves

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Abstract
A general scheme for determining and studying integrable deformations of algebraic curves, based on the use of Lenard relations, is presented. The method is illustrated with the analysis of the hyperelliptic case. An associated multi-Hamiltonian hierarchy of systems of hydrodynamic type is characterized.

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1. Introduction

Algebraic curves arise in the study of various problems for nonlinear differential equations. The theory of finite-gap solutions of integrable equations and the Whitham averaging theory are, probably, the best known examples of the relevance of algebraic curves and their deformations in the context of nonlinear integrable models (see, e.g., [1–3]).

During the seventies it was shown (see, e.g., [1, 2]) that the construction of the finite-gap solutions of the Korteweg–de Vries (KdV) equation and other soliton equations is closely associated with hyperelliptic curves

$$p^2 = \prod_{j=1}^{2g+1} (z - z_j), \tag{1}$$

where the parameters z_j (branch points) represent the boundary values E_j of the energy of the non-degenerate gaps. While the standard isospectral symmetries of the KdV equation leave E_j invariant, the non-isospectral symmetries change the values E_j [4, 5] and, consequently, they deform the curve (1).

The Whitham averaging of the finite-gap solutions of the KdV equation also leads to deformations of hyperelliptic curves. In this case, the branch points $\mathbf{z} := (z_1, \ldots, z_{2g+1})$ evolve according to a hydrodynamic system of diagonal type [6]

$$\frac{\partial z_j}{\partial t} = \Omega(\mathbf{z}) \frac{\partial z_j}{\partial x}, \tag{2}$$

where x, t are *slow variables*. Similar deformations of algebraic curves are produced in the application of the Whitham averaging method to other soliton equations [7].

A general theory describing the deformations arising in the Whitham averaging method in terms of dynamical systems (Whitham hierarchies) was provided by Krichever [8–10]. It is based on flows defined on spaces of *algebraic-geometric data*

$$\widehat{M}_{g,N} = \{\mathcal{C}_g, P_\alpha, k_\alpha^{-1}(P), \alpha = 1, \dots, N\}, \tag{3}$$

where $\mathcal{C}_g$ is an algebraic curve of genus g and $k_\alpha^{-1}(P)$ are local coordinates in the neighbourhoods of N points $P_\alpha \in \mathcal{C}_g$, $\big(k_\alpha^{-1}(P_\alpha) = 0\big)$. One important feature of the Whitham flows is the presence of *quasiclassical Lax pairs*. Thus the data k_α evolve in the form

$$\frac{\partial k_\alpha}{\partial t} = \{Q, k_\alpha\} := \frac{\partial Q}{\partial p}\frac{\partial k_\alpha}{\partial x} - \frac{\partial Q}{\partial x}\frac{\partial k_\alpha}{\partial p}. \tag{4}$$

Here the variable $p := \int^P \mathrm{d}Q_1$ is defined through a meromorphic form $\mathrm{d}Q_1$ holomorphic outside P_1 and such that near P_1 behaves as $\mathrm{d}Q_1 = \mathrm{d}\big(k_1 + \mathcal{O}\big(k_1^{-1}\big)\big)$. By expressing $p = p(k_\alpha, x, t)$ as a function of any of the coordinates k_α the Lax equations (4) reduce to

$$\frac{\partial p}{\partial t} = \frac{\partial Q}{\partial x}. \tag{5}$$

This formulation leads at once to the existence of an *action function* $S = S(p, x, t)$ such that in a neighbourhood U_α of each point P_α

$$p = \frac{\partial S}{\partial x}, \qquad Q = \frac{\partial S}{\partial t}, \tag{6}$$

where $p = p(k_\alpha, x, t)$. For $g \neq 0$ the Whitham hierarchies induce deformations of $\mathcal{C}_g$. Moreover, the equations describing the motion of the moduli of $\mathcal{C}_g$ turn out to be diagonal systems of hydrodynamic type.

There is another type of deformations of algebraic curves which is associated with the so-called *algebraic orbits* [9] of the Whitham hierarchies for $g = 0$. For example, let us consider the zero-genus case of the Whitham hierarchy with $N = 1$. It constitutes the Zabolotskaya–Khokhlov dispersionless KP (dKP) hierarchy

$$\frac{\partial k}{\partial t_n} = \{Q_n, k\}, \qquad n \geqslant 1, \tag{7}$$

where

$$k = p + \sum_{n=1}^{\infty} \frac{a_n(x, t)}{p^n}, \qquad Q_n := (k^n)_{\geqslant 0}, \qquad t := (t_1, t_2, \dots). \tag{8}$$

Algebraic orbits of (7) are deformations of algebraic curves

$$f(k) = E(p, x, t), \tag{9}$$

where $E = E(p, x, t)$ is a meromorphic function of p, such that (9) determines a reduction of the hierarchy (8). Well-known examples are

(1) Gelfand–Dikii reductions

$$k^N = p^N + u_{N-2}p^{N-2} + \cdots + u_0,$$

(2) Zakharov reductions

$$k = p + \sum_{i=1}^{M} \frac{h_i}{p - v_i},$$

(3) Kodama–Krichever reductions

$$k^{N+1} = p^{N+1} + u_1 p^{N-1} + \cdots + u_N + \frac{v_1}{p - v_0} + \cdots + \frac{v_M}{(p - v_0)^M}.$$

The function E in (22) depends on the variables (x, t) through a finite number of functions (for instance $(u_0, \ldots, u_{N-2})$ in the Gelfand–Dikii case) and, as a consequence of (7), these functions turn out to evolve according to a system of hydrodynamic type.

Algebraic orbits appear in a natural way within the *quasiclassical* (dispersionless) limit of integrable systems [11–20]. For scalar Lax-type equations the quasiclassical limit is determined by the leading order of the expansion arising from inserting in the associated spectral problem the ansatz

$$\psi = \exp\left(\frac{S}{\epsilon}\right), \qquad \epsilon \to 0,$$

and substituting $\partial_x \to \epsilon \partial_x$.

The present work is motivated by the following pair of observations:

1. The classification and analysis of reductions of the dKP hierarchy is an interesting subject still far from being completed [21, 22]. An obvious related question is to investigate the particular class of reductions provided by the algebraic orbits. A recent work by Kokotov and Korotkin [23] shows the relevance of the theory of Hurwitz spaces to deal with these problems.
2. Deformations of algebraic curves similar to the dKP algebraic orbits have been formulated which do not correspond to the standard dispersionless hierarchies. Indeed, the integrable hierarchy associated with the *energy-dependent* Schrödinger problem [24] admits a dispersionless limit which leads to a family of deformations of the curves [20]

$$p^2 = k^{2N+1} + \sum_{n=1}^{N} u_n k^{2n-1}. \tag{10}$$

These curves do not constitute any reduction of the dKP hierarchy and, as it is shown in [25], they should be understood in terms of the singular sectors of a dKdV Grassmannian structure.

In this work, we propose a method to attack frontally the problem of analysing and classifying integrable deformations of algebraic curves ruled by systems of hydrodynamic type, without relying on any particular type of dispersionless hierarchy. Thus, we consider general algebraic curves $\mathcal{C}$ defined by polynomial equations

$$F(p, k) := p^N - \sum_{n=0}^{N-1} u_n(k) p^n = 0, \qquad u_n \in \mathbb{C}[k], \tag{11}$$

and investigate deformations $\mathcal{C}(x, t)$ consistent with the degrees of the polynomials u_n and which, similar to [8, 9], are characterized by the existence of an *action* function $\mathbf{S} = \mathbf{S}(k, x, t)$ verifying

1. The multiple-valued function $\boldsymbol{p} = \boldsymbol{p}(k)$ determined by (11) can be expressed as

$$\boldsymbol{p} = \mathbf{S}_x.$$

2. The function $\mathbf{S}_t$ represents, like $\boldsymbol{p} = \mathbf{S}_x$, a meromorphic function on $\mathcal{C}(x, t)$ with poles only at $k = \infty$.

As a consequence of these conditions $\boldsymbol{p}$ obeys the equation

$$\partial_t \boldsymbol{p} = \partial_x \boldsymbol{Q}, \tag{12}$$

where $\boldsymbol{Q} := \mathbf{S}_t$ is assumed to be of the form

$$\boldsymbol{Q} = \sum_{r=0}^{N-1} a_r(k)\boldsymbol{p}^r, \qquad a_r \in \mathbb{C}[k].$$

We take equations (11) and (12) as our starting point and express the coefficients u_n of (11) as elementary symmetric functions of the branches p_i of $\boldsymbol{p}$ (Viète theorem) to formulate the corresponding system of evolution equations for u_n. They turn out to admit a simple general form in terms of symmetric functions of p_i involving the so-called *power sums* functions. The requirement of consistency with the polynomial character of the coefficients u_n as functions of k imposes severe constraints to the degrees of the polynomials $u_n(k)$ in (11). To deal with the problem of characterizing consistent deformations, we develop a technique based on solving Lenard-type relations, which can be viewed as a quasiclassical version of the fruitful *resolvent method* [25] of the theory of Lax pairs. Thus, we provide a class of consistent deformations determined by systems of hydrodynamic type for the coefficients u_n. Moreover, they have a natural associated set of Riemann invariants. The analysis of the completeness of these sets of Riemann invariants is found to be related to the existence of gauge symmetries. Finally we note that the branches p_i of the function $\boldsymbol{p}$ are the basic dependent variables in our approach.

To illustrate our analysis we study in detail the hyperelliptic case

$$p^2 - v(k)p - u(k) = 0, \qquad u, v \in \mathbb{C}[k].$$

We present a class of quasiclassical deformations of these curves determined by a hierarchy of compatible multi-Hamiltonian systems of hydrodynamic type which is analysed from the R-matrix point of view. A quantum (dispersionful) counterpart of this hierarchy is also discussed. The analysis of the deformations of both the general case of (11) and its reductions will be presented elsewhere.

2. Algebraic curves

We start with some notation conventions to enounce the results of algebraic geometry [26–28] which are particularly helpful in our analysis.

Let $\mathcal{C}$ denote an algebraic curve determined by (11). Its associated function $\boldsymbol{p} = \boldsymbol{p}(k)$ describes a multiple-valued function determined by N branches $p_i = p_i(k)$ $(i = 0, \ldots, N-1)$ satisfying

$$F(p, k) = \prod_{i=0}^{N-1}(p - p_i(k)). \tag{13}$$

We denote by $\mathbb{C}((k))$ the field of power series in k with at most a finite number of terms with positive powers

$$\sum_{n=-\infty}^{N} a_n k^n, \qquad N \in \mathbb{Z}.$$

The following general result [26, 27] will be used in our subsequent analysis:

Theorem 1 (Newton theorem)**.** *There exists a positive integer l such that the N branches*

$$p_i(z) := (p_i(k))|_{k=z^l}, \tag{14}$$

are elements of $\mathbb{C}((z))$*. In other words, they are Laurent series of finite order as* $z \to \infty$

$$p_i(z) = \sum_{n=0}^{N_i} a_n^{(i)} z^n + \sum_{n=1}^{\infty} \frac{b_n^{(i)}}{z^n}, \qquad i = 1, \ldots, N.$$

Furthermore, if $F(p,k)$ *is irreducible as a polynomial over the field* $\mathbb{C}((k))$ *then* $l_0 = N$ *is the least permissible* l *and the branches* $p_i(z)$ *can be labelled so that*

$$p_i(z) = p_0(\epsilon^i z), \qquad \epsilon := \exp \frac{2\pi \mathrm{i}}{N}.$$

Notation convention: Henceforth, given an algebraic curve $\mathcal{C}$ *we will denote by* z *the variable associated with the least positive integer* l_0 *for which the substitution* $k = z^{l_0}$ *implies* $p_i \in \mathbb{C}((z)), \forall i$*.*

Example. The curves

$$p^2 - u(k) = 0, \qquad u(k) := \sum_{i=0}^{m} u_i k^i, \qquad u_m \neq 0,$$

have the branches

$$p_\pm := \sqrt{u(k)} = \sqrt{u_m k^m}\left(1 + \mathcal{O}\left(\frac{1}{k}\right)\right), \qquad k \to \infty,$$

so that $F := p^2 - u(k)$ is an irreducible (reducible) polynomial over $\mathbb{C}((k))$ for odd (even) m and

$$k = z, \quad \text{for even } m, \qquad k = z^2, \quad \text{for odd } m.$$

According to Viète theorem [29] we may write the coefficients u_n of (11) in terms of the branches p_i as

$$u_n = (-1)^{N-n-1} \mathrm{s}_{N-n}, \tag{15}$$

where $\mathrm{s}_k = \mathrm{s}_k(p_0, \ldots, p_{N-1})$ are the elementary symmetric polynomials

$$\mathrm{s}_k = \sum_{0 \leqslant i_1 < \cdots < i_k \leqslant N-1} p_{i_1} \cdots p_{i_k}.$$

In our study, we will use also the so-called *power sums* [29]

$$\mathcal{P}_k = p_0^k + \cdots + p_{N-1}^k, \qquad k \geqslant 0.$$

These symmetric functions are polynomials in the elementary symmetric functions s_k and, consequently, they can be written as polynomials in the coefficients u_n. In order to obtain these polynomials, one can use Newton recurrence formulae [29]

$$\begin{aligned} \mathcal{P}_k &= k u_{N-k} + u_{N-k+1}\mathcal{P}_1 + \cdots + u_{N-1}\mathcal{P}_{k-1}, && 1 \leqslant k \leqslant N, \\ \mathcal{P}_k &= u_0 \mathcal{P}_{k-N} + u_1 \mathcal{P}_{k-N+1} + \cdots + u_{N-1}\mathcal{P}_{k-1}, && k > N, \end{aligned} \tag{16}$$

as well as the explicit determinant expressions [30]

$$\mathcal{P}_k = \begin{vmatrix} u_{N-1} & 1 & 0 & \cdots & 0 \\ -2u_{N-2} & u_{N-1} & 1 & \cdots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ (-1)^k (k-1) u_{N-k+1} & \multicolumn{3}{c}{\dots\dots\dots\dots} & 1 \\ (-1)^{k+1} k u_{N-k} & \multicolumn{3}{c}{\dots\dots\dots\dots} & u_{N-1} \end{vmatrix}, \tag{17}$$

where it is assumed that $u_N := -1$, $u_n := 0$, $n > N$. One has also a similar determinant formula for u_n in terms of $\mathcal{P}_k$

$$u_n = -\frac{(-1)^{N-n}}{(N-n)!}\begin{vmatrix} \mathcal{P}_1 & 1 & 0 & \cdots & 0 \\ \mathcal{P}_2 & \mathcal{P}_1 & 2 & \cdots & 0 \\ \multicolumn{5}{c}{\dotfill} \\ \mathcal{P}_{N-n-1} & \ldots\ldots & & \mathcal{P}_1 & N-n-2 \\ \mathcal{P}_{N-n} & \ldots\ldots\ldots\ldots & & & \mathcal{P}_1 \end{vmatrix}, \tag{18}$$

where $0 \leqslant n \leqslant N-1$.

3. Quasiclassical deformations of algebraic curves

Let us consider the problem of characterizing families $\mathcal{C}(x,t)$ $(x, t \in \mathbb{C})$ of algebraic curves

$$F := p^N - \sum_{n=0}^{N-1} u_n(k, x, t) p^n = 0, \qquad u_n \in \mathbb{C}[k], \tag{19}$$

which admit a multiple-valued function $\mathbf{S} = \mathbf{S}(k, x, t)$ (*action function*) verifying

$$\boldsymbol{p} = \mathbf{S}_x.$$

Furthermore, we assume that

$$\boldsymbol{Q} := \mathbf{S}_t,$$

is, like $\boldsymbol{p}$, a meromorphic function on $\mathcal{C}(x,t)$ which has poles at $k = \infty$ only. Thus we assume that $\boldsymbol{Q}$ can be expressed in the form

$$\boldsymbol{Q} = \sum_{r=0}^{N-1} a_r(k) \boldsymbol{p}^r, \qquad a_r \in \mathbb{C}[k]. \tag{20}$$

The coefficients a_r, and consequently the function $\boldsymbol{Q}$, will be explicitly dependent on x and t also.

Thus the deformations of (19) satisfying these requirements are characterized by equations of the form

$$\partial_t \boldsymbol{p} = \partial_x \left(\sum_{r=0}^{N-1} a_r(k) \boldsymbol{p}^r \right), \qquad a_r \in \mathbb{C}[k]. \tag{21}$$

We will refer to the flows of the form (21) as *quasiclasssical deformations* of the curve (19). They are directly connected to Lax equations of quasiclassical type. To see this property note that in terms of the branches of $\boldsymbol{p}$, the flow (21) reduces to the system

$$\begin{aligned} &\partial_t p_i = \partial_x Q_i, \qquad i = 0, \ldots, N-1 \\ &Q_i := \sum_{r=0}^{N-1} a_r(k) p_i^r. \end{aligned} \tag{22}$$

These equations imply the existence of N branches $\mathrm{S}_i = \mathrm{S}_i(z, \cdot, \cdot) \in \mathbb{C}((z))$ of $\mathbf{S}$ verifying

$$\mathrm{dS}_i = p_i \,\mathrm{d}x + Q_i \,\mathrm{d}t + m_i \,\mathrm{d}z, \qquad m_i := \frac{\partial \mathrm{S}_i}{\partial z},$$

where $k = z^{l_0}$. Hence we have

$$\mathrm{d}p_i \wedge \mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}t = \mathrm{d}z_i \wedge \mathrm{d}m_i, \tag{23}$$

where $z_i = z(p_i, x, t)$, $i = 0, \ldots, N-1$ stand for the functions obtained by inverting $p_i = p_i(z, x, t)(i = 0, \ldots, N-1)$. We now consider the change of variables

$$(p_i, x, t) \mapsto (z_i, x, t),$$

and use the standard technique of [13]. Thus by identifying the coefficients of $\mathrm{d}p_i \wedge \mathrm{d}x$, $\mathrm{d}p_i \wedge \mathrm{d}t$ and $\mathrm{d}x \wedge \mathrm{d}t$ in (23), we find

$$\frac{\partial z_i}{\partial p_i}\frac{\partial m_i}{\partial x} - \frac{\partial z_i}{\partial x}\frac{\partial m_i}{\partial p_i} = 1, \qquad \frac{\partial z_i}{\partial p_i}\frac{\partial m_i}{\partial t} - \frac{\partial z_i}{\partial t}\frac{\partial m_i}{\partial p_i} = \frac{\partial Q_i}{\partial p_i},$$
$$\frac{\partial z_i}{\partial x}\frac{\partial m_i}{\partial t} - \frac{\partial z_i}{\partial t}\frac{\partial m_i}{\partial x} = \frac{\partial Q_i}{\partial x},$$

so that the functions z_i satisfy the quasiclassical Lax equations

$$\frac{\partial z_i}{\partial t} = \frac{\partial Q_i}{\partial p_i}\frac{\partial z_i}{\partial x} - \frac{\partial Q_i}{\partial x}\frac{\partial z_i}{\partial p_i}, \qquad i = 0, \ldots, N-1. \tag{24}$$

Our next step is to characterize the evolution law of the coefficients u_n induced by (21). From (22) it follows:

$$\frac{\partial u_n}{\partial t} = \sum_{i,j}\frac{\partial u_n}{\partial p_i}\partial_x\big(a_j p_i^j\big).$$

At this point it is important to use the following identities

$$\frac{\partial u_n}{\partial p_i} = p_i^{N-n-1} - \sum_{m=n+1}^{N-1} u_m p_i^{m-n-1}, \tag{25}$$

which derive from (11) by differentiating with respect to p_i and identifying coefficients of powers of p in the resulting equation

$$\frac{F}{p - p_i} = \sum_n \frac{\partial u_n}{\partial p_i} p^n.$$

Thus, one deduces at once that in terms of the variables u_n, the flow (21) reduces to the system

$$\partial_t \boldsymbol{u} = J_0 \boldsymbol{a}, \tag{26}$$

where we are denoting

$$\boldsymbol{u} := \begin{pmatrix} u_{N-1} \\ \vdots \\ u_0 \end{pmatrix}, \qquad \boldsymbol{a} := \begin{pmatrix} a_0 \\ \vdots \\ a_{N-1} \end{pmatrix},$$

and J_0 is the matrix differential operator

$$J_0 := T^\top V^\top \partial_x V, \tag{27}$$

where

$$T := \begin{pmatrix} 1 & -u_{N-1} & \cdots & -u_1 \\ 0 & 1 & \cdots & -u_2 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & 1 \end{pmatrix}, \tag{28}$$

and V is the Vandermonde matrix

$$V := \begin{pmatrix} 1 & p_{N-1} & \cdots & p_{N-1}^{N-1} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 1 & p_0 & \cdots & p_0^{N-1} \end{pmatrix}. \tag{29}$$

The explicit expression of the operator J_0 in terms of the coefficients u_n follows from the observation that the matrix elements of $V^\top \partial_x V$ can be written as

$$(V^\top \partial_x V)_{ij} = \mathcal{P}_{i+j}\partial_x + \frac{j}{i+j}\mathcal{P}_{i+j,x}, \qquad 0 \leqslant i, j \leqslant N-1, \tag{30}$$

where $\mathcal{P}_k(p_0, \ldots, p_{N-1})$ are the power sums of the variables p_i. Hence, the system (26) becomes ($u_N := -1$)

$$\partial_t u_n = -\sum_{r=0}^{N-n-1}\sum_{m=0}^{N-1} u_{n+r+1}\left(\mathcal{P}_{m+r}\partial_x + \frac{m}{m+r}\mathcal{P}_{m+r,x}\right)a_m. \tag{31}$$

Example 1. For $N = 2$, the equation for the curve is

$$F := p^2 - u_1 p - u_0 = 0, \tag{32}$$

and the first power sums are

$$\mathcal{P}_1 = u_1, \qquad \mathcal{P}_2 = u_1^2 + 2u_0.$$

Thus we find

$$J_0 = \begin{pmatrix} 2\partial_x & \partial_x(u_1\cdot) \\ -u_1\partial_x & 2u_0\partial_x + u_{0,x} \end{pmatrix}. \tag{33}$$

Example 2. For $N = 3$,

$$F := p^3 - u_2 p^2 - u_1 p - u_0 = 0, \tag{34}$$

and

$$\mathcal{P}_1 = u_2, \qquad \mathcal{P}_2 = 2u_1 + u_2^2,$$
$$\mathcal{P}_3 = 3u_0 + 3u_1u_2 + u_2^3,$$
$$\mathcal{P}_4 = 4u_0u_2 + 2u_1^2 + 4u_1u_2^2 + u_2^4,$$

$$J_0 = \begin{pmatrix} 3\partial_x & u_2\partial_x + u_{2,x} & \left(2u_1 + u_2^2\right)\partial_x + \left(2u_1 + u_2^2\right)_x \\ -2u_2\partial_x & 2u_1\partial_x + u_{1,x} & (3u_0 + u_1u_2)\partial_x + 2u_{0,x} + 2u_1u_{2,x} \\ -u_1\partial_x & 3u_0\partial_x + u_{0,x} & u_0u_2\partial_x + 2u_0u_{2,x} \end{pmatrix}. \tag{35}$$

3.1. Consistency conditions and Lenard relations

Our main aim is to determine expressions for $\boldsymbol{a}$ depending on z and $\boldsymbol{u}$ such that (31) is consistent with the polynomial dependence of $\boldsymbol{u}$ on the variable k. That is to say, if $d_n := \mathrm{degree}(u_n)$ are the degrees of the coefficients u_n as polynomials in k, then (31) must satisfy

$$\mathrm{degree}(J_0\boldsymbol{a})_n \leqslant d_n, \qquad \forall n.$$

In the case of consistency, (31) will provide a system of hydrodynamic type for the coefficients of the polynomials u_n and, as a consequence of (22), the coefficients of the expansions of the branches

$$p_i(z) = \sum_{n=0}^{N_i} h_n^{(i)}(\boldsymbol{u})z^n + \sum_{n=1}^{\infty}\frac{h_n^{(i)}(\boldsymbol{u})}{z^n}, \qquad i = 1, \ldots, N, \quad k = z^l,$$

are conserved densities. Our main strategy for finding consistent systems is based on using Lenard type relations

$$J_0\boldsymbol{R} = 0, \qquad \boldsymbol{R} := (R_1, \ldots, R_N)^\top, \qquad R_i \in \mathbb{C}((k)), \tag{36}$$

and then considering systems of the form

$$\boldsymbol{u}_t = J_0 \boldsymbol{a}, \qquad \boldsymbol{a} := \boldsymbol{R}_+. \tag{37}$$

Here $(\cdot)_+$ and $(\cdot)_-$ indicate the parts of non-negative and negative powers in k, respectively. In these cases from the identity

$$J_0 \boldsymbol{a} = J_0 \boldsymbol{R}_+ = -J_0 \boldsymbol{R}_-,$$

it is clear that a sufficient condition for the consistency of (37) is that

$$\operatorname{degree}(J_0)_{nm} \leqslant d_n + 1, \tag{38}$$

for all n and all m such that $a_m = (\boldsymbol{R}_+)_m \neq 0$.

If we impose (38) for all $0 \leqslant n, m \leqslant N-1$, we get a sufficient condition for consistency which only depends on the curve (11) and does not refer to the particular solution of the Lenard relation involved in (37).

Examples. From (33) it is straightforward to see that for $N = 2$ the systems of the form (37) verifying (38) for all $0 \leqslant n, m \leqslant 1$ are characterized by the constraint

$$d_1 \leqslant d_2 + 1. \tag{39}$$

For $N = 3$ if we impose (38) for all $0 \leqslant n, m \leqslant 2$ then from (35) we get the constraints

$$\begin{aligned} &d_2 \leqslant 1, \qquad &&d_1 \leqslant d_2 + 1, \qquad &&d_2 \leqslant d_1 + 1, \\ &d_0 \leqslant d_1 + 1, \qquad &&d_1 \leqslant d_0 + 1, \end{aligned} \tag{40}$$

which lead to the following 13 nontrivial choices for (d_0, d_1, d_2):

$$\begin{aligned} &(1,0,0), (0,1,0), (1,1,0), (2,1,0), \\ &(0,0,1), (1,0,1), (0,1,1), (1,1,1) \\ &(2,1,1), (0,2,1), (1,2,1), (2,2,1), \\ &(3,2,1). \end{aligned}$$

Examples of flows (37) in which some components of $\boldsymbol{a}$ vanish identically arise by imposing reduction conditions to the curve (11). For instance, let us consider the curve [19]

$$p^N - u_0(k) = 0, \qquad u_0 \in \mathbb{C}[k].$$

The branches of $\boldsymbol{p}$ are

$$p_i(k) = \epsilon^i p_0(k), \qquad p_0(k) := \sqrt[N]{u_0(k)}, \qquad \epsilon := \exp\left(\frac{2\pi}{N}\mathrm{i}\right), \tag{41}$$

with $\sqrt[N]{u_0(k)}$ being a given Nth root of $u_0(k)$. The systems (22) which are compatible with (41) are those of the form

$$\partial_t p_i = \partial_x (a_1(k) p_i),$$

so that the corresponding evolution law for u_0 is

$$\partial_t u_0 = J_0 a_1, \qquad J_0 := N p_0^{N-1} \partial_x (p_0 \cdot) = N u_o \partial_x + u_{0,x}.$$

Thus, if d denotes the degree of $u_0(k)$, we get the following solutions of the Lenard relation $J_0 R = 0$ in $\mathbb{C}((k))$:

$$R_M = \frac{k^{M+\frac{d}{N}}}{p_0(k)} = \frac{k^{M+\frac{d}{N}}}{\sqrt[N]{u_0(k)}}, \qquad M \geqslant 0.$$

They determine the following infinite set of consistent systems for any degree d of $u_0(k)$

$$\partial_{t_M} u_0 = (N u_0 \partial_x + u_{0,x}) \left(\frac{k^{M+\frac{d}{N}}}{\sqrt[N]{u_0(k)}} \right)_+ .$$

Let us consider now the general case of (11). There is a natural class of solutions of the Lenard relations (36). Indeed, from the expression (27) of J_0 it is obvious that for any constant vector $\boldsymbol{c} \in \mathbb{C}^N$

$$\boldsymbol{R} := V^{-1}\boldsymbol{c},$$

is a solution of (36). Furthermore, according to (25)

$$VT = \left(\frac{\partial(u_{N-1}, \ldots, u_0)}{\partial(p_{N-1}, \ldots, p_0)} \right)^\top ,$$

so that

$$V^{-1} = T \left(\frac{\partial(p_{N-1}, \ldots, p_0)}{\partial(u_{N-1}, \ldots, u_0)} \right)^\top . \tag{42}$$

In this way, we have a set of basic solutions $\boldsymbol{R}_i = V^{-1}\boldsymbol{c}_i, (\boldsymbol{c}_i)_j := \delta_{ij}$ of the Lenard relations given by

$$\boldsymbol{R}_i := T \left(\frac{\partial p_i}{\partial u_{N-1}} \cdots \frac{\partial p_i}{\partial u_0} \right)^\top , \qquad i = 0, \ldots, N-1. \tag{43}$$

We note that direct implicit differentiation of (11) gives

$$\frac{\partial p_i}{\partial u_j} = \frac{p_i^j}{F_p(p_i)}. \tag{44}$$

In summary, systems of the form

$$\partial_t \boldsymbol{u} = J_0 (T \nabla C)_+, \tag{45}$$

where

$$C = \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x \sum_i f_i(z) p_i, \qquad f_i \in \mathbb{C}[z], \qquad \nabla C = \left(\frac{\delta C}{\delta u_{N-1}} \cdots \frac{\delta C}{\delta u_0} \right)^\top ,$$

with γ being the unit circle in $\mathbb{C}$, are consistent provided the following conditions are satisfied,

$$\mathrm{degree}(J_0)_{nm} \leqslant d_n + 1, \tag{46}$$

$$\frac{\delta C}{\delta u_n} \in \mathbb{C}((z^l)), \tag{47}$$

for all $0 \leqslant n, m \leqslant N-1$.

3.2. Gelfand–Dikii flows

Let us show how the systems (45) include the quasiclassical versions of the Gelfand–Dikii hierarchies. To this end, we consider curves (11) of the form

$$F := p^N - \sum_{n=1}^{N-1} u_n(x, t) p^n - v_0(x) - z^N = 0.$$

They give rise to a matrix T which is z-independent, so that the corresponding systems (45) become

$$\partial_t \boldsymbol{u} = J(\nabla C)_+, \tag{48}$$

where $J := J(z, \boldsymbol{u})$ is the symplectic operator

$$J := T^\top V^\top \partial_x V T \tag{49}$$

$$= \left(\frac{\partial(u_{N-1}, \ldots, u_0)}{\partial(p_{N-1}, \ldots, p_0)}\right) \partial_x \left(\frac{\partial(u_{N-1}, \ldots, u_0)}{\partial(p_{N-1}, \ldots, p_0)}\right)^\top, \tag{50}$$

$$J_{ij} = \sum_{l=0}^{i} \sum_{m=0}^{j} u_{N+l-i} \left(\mathcal{P}_{l+m}\partial_x + \frac{m}{l+m}\mathcal{P}_{l+m,x}\right) u_{N+m-j}, \tag{51}$$

where we are denoting $u_N := -1$.

From (16) one proves at once that for $N \leqslant i \leqslant 2N-1$ the functions $\mathcal{P}_i$ are linear in z^N

$$\mathcal{P}_i = z^N \mathcal{P}_{1,i} + \mathcal{P}_{2,i},$$

and

$$T \begin{pmatrix} \mathcal{P}_{1,2N-1} \\ \vdots \\ \mathcal{P}_{1,N} \end{pmatrix} = \begin{pmatrix} \mathcal{P}_{N-1} \\ \vdots \\ \mathcal{P}_0 \end{pmatrix}.$$

Hence, as a consequence it follows that J is of the form

$$J(z, \boldsymbol{u}) = z^N J_1 + J_2.$$

In particular the operator J_1 is given by ($u_N := -1$)

$$(J_1)_{ij} = \begin{cases} -((2N-i-j)u_{2N-i-j}\partial_x + (N-j)u_{2N-i-j,x}) & \text{if } i+j \geqslant N \\ 0 & \text{otherwise.} \end{cases}$$

The operators J_1 and J_2 form a pair of compatible symplectic operators which describe the quasiclassical limits of the Gelfand–Dikii symplectic operators for the standard hierarchies of scalar Lax pairs [25].

Let us denote by p_0 the branch of $\boldsymbol{p}$ such that

$$p_0 = z + \frac{h_0(\boldsymbol{u})}{z} + \cdots \frac{h_n(\boldsymbol{u})}{z^n} + \cdots, \qquad z \to \infty. \tag{52}$$

The Nth dispersionless Gelfand–Dikii hierarchy can be formulated as the system of flows

$$\frac{\partial z}{\partial t_M} = \frac{\partial Q_M}{\partial p_0}\frac{\partial z_i}{\partial x} - \frac{\partial Q_M}{\partial x}\frac{\partial z}{\partial p_0}, \qquad M \geqslant 1, \tag{53}$$

where $z := z(p_0, x, \boldsymbol{t})$ and

$$Q_M(p_0, x, \boldsymbol{t}) := (z^M)_\oplus, \qquad \boldsymbol{t} = (t_1, \ldots, t_n, \ldots).$$

Here $(\cdot)_\oplus$ and $(\cdot)_\ominus$ denote the parts of non-negative and negative powers in p_0, respectively. It is straightforward to deduce that the Gelfand–Dikii herarchy (53) corresponds to the system of flows

$$\partial_{t_M} \boldsymbol{u} = J(\nabla C_M)_+, \tag{54}$$

where the functionals C_M are given by

$$C_M[\boldsymbol{u}] := \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} \int \mathrm{d}x \sum_j (\epsilon^j z)^M p_0(\epsilon^j z).$$

Alternatively, due to the Lenard relation

$$J(\nabla C_M)_+ = -J(\nabla C_M)_-,$$

we have

$$\partial_{t_M} \boldsymbol{u} = N J_2(\nabla h_M) = -N J_1(\nabla h_{M+N}). \tag{55}$$

3.3. Riemann invariants and gauge transformations

From (24) it follows that the values $z_{is} := z_i(p_s, \boldsymbol{u})$ corresponding to points $p_{i,s}$ at which

$$\frac{\partial z_i}{\partial p}(p_{i,s}, \boldsymbol{u}) = 0$$

are Riemann invariants of the hydrodynamic system (31). Therefore an important question is to know under what conditions these Riemann invariants are sufficient to integrate (31). It turns out that this problem is closely related to the analysis of the quasiclassical version of gauge transformations of integrable systems [31].

By a gauge transformation of a consistent system (21) we mean a map $\boldsymbol{p} \to \boldsymbol{p} + g$, $g = g(z, x, t)$

$$p_i \to p_i + g, \qquad g \in \mathbb{C}[z], \qquad \forall i,$$

such that the induced transformation on the coefficients u_n

$$u_n \to \sum_{r=0}^{N-n} \binom{n+r}{r} u_{n+r} g^r,$$

preserves the degrees d_n of u_n as polynomials in z^l.

Gauge transformations possess an obvious set of $N-1$ independent invariants

$$w_i := p_i - p_{i+1}, \qquad i = 0, \ldots, N-2,$$

and a *gauge variable*

$$\rho := \frac{1}{N} u_{N-1} = \frac{1}{N} \sum_i p_i, \qquad \rho \to \rho + g.$$

Like in the dispersionful case [31], we may describe the dynamical variables u_n in terms of $N-1$ gauge invariants and the gauge variable ρ. As far as the Riemann invariants $z_{is} = z_{is}(\boldsymbol{u})$ are concerned we observe that they satisfy

$$R(z_{is}) = 0,$$

where $R(z)$ is the discriminant of F (the resultant of the function F and its derivative F_p with respect to p). It is given by [32]

$$R(z) = (-1)^{N(N-1)/2} \prod_{i>j} (p_i(z) - p_j(z))^2,$$

which is obviously a gauge invariant. Hence we conclude that the Riemann invariants $z_{is} = z_{is}(\boldsymbol{u})$ are gauge invariants. Therefore, they cannot describe degrees of freedom associated with gauge variables.

4. Deformations of hyperelliptic curves

The hyperelliptic curves are characterized by quadratic equations in p

$$F := p^2 - v(k,x)p - u(k,x) = 0, \tag{56}$$

where

$$v = \sum_{i=0}^{d_1} v_i k^i, \qquad u = \sum_{i=0}^{d_0} u_i k^i.$$

The branches of p are given by

$$p_\pm = \tfrac{1}{2}\big(v \pm \sqrt{v^2 + 4u}\big), \tag{57}$$

and the map $(p_+, p_-) \mapsto (v, u)$ is

$$v = p_+ + p_-, \qquad u = -p_+ p_-. \tag{58}$$

Let us denote

$$d := \max(2d_1, d_0).$$

Note that F is irreducible (reducible) over $\mathbb{C}((k))$ for d odd (even). Thus we set

$$k = z, \quad \text{for } d \text{ even}, \qquad k = z^2, \quad \text{for } d \text{ odd}. \tag{59}$$

According to (45) we consider equations of the form

$$\partial_t \boldsymbol{u} = J_0(T\nabla C)_+, \tag{60}$$

where

$$\boldsymbol{u} := \begin{pmatrix} v \\ u \end{pmatrix}, \qquad J_0(v,u) = \begin{pmatrix} 2\partial_x & \partial_x(v\cdot) \\ -v\partial_x & 2u\partial_x + u_x \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix},$$

for functionals

$$C = \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x\,(f_+(z)p_+ + f_-(z)p_-),$$

As we have seen above these systems are consistent provided

$$d_1 \leqslant d_0 + 1,$$

and

$$\nabla C \in \mathbb{C}((z^2)), \qquad \text{if } d \text{ is odd.}$$

By direct calculation it is straightforward to check that

$$T(\nabla C^{(\pm)}) = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \nabla C^{(\mp)}, \qquad C^{(\pm)} := \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x\; p_\pm.$$

Hence, without loss of generality, we can generate the set of flows (60) from the hierarchy

$$\partial_{t_N} \boldsymbol{u} = J(\nabla C_N)_+, \qquad N \geqslant 0, \tag{61}$$

where $J = J(u, v)$ is the operator

$$J := \begin{pmatrix} -2\partial_x & \partial_x(v\cdot) \\ v\partial_x & 2u\partial_x + u_x \end{pmatrix} \tag{62}$$

and

$$\begin{aligned} C_N[v,u] &:= \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x\, \frac{z^N}{2}(p_+ - p_-) \\ &= \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x\, \frac{z^N}{2}\sqrt{v^2 + 4u}, \end{aligned} \tag{63}$$

with N being odd when d is odd.

Observe that

$$\nabla C_N = \begin{pmatrix} \frac{\delta C_N}{\delta v} \\ \frac{\delta C_N}{\delta u} \end{pmatrix}_+ = \begin{pmatrix} \frac{z^N}{2} \frac{v}{\sqrt{v^2+4u}} \\ \frac{z^N}{\sqrt{v^2+4u}} \end{pmatrix}_+ .$$

4.1. R-matrix theory and multi-Hamiltonian structure

Equations (61) represent a hierarchy of compatible multi-Hamiltonian systems associated with a R-matrix structure. In order to describe this property we introduce the Lie algebra $\mathcal{G}$ with elements

$$\boldsymbol{\alpha}(z, x) = \begin{pmatrix} \alpha_1(z, x) \\ \alpha_2(z, x) \\ \alpha_3(x) \end{pmatrix}, \qquad \alpha_i(\cdot, x) \in \mathbb{C}((z)), \qquad i = 1, 2,$$

and commutator defined by

$$[\boldsymbol{\alpha}, \boldsymbol{\beta}] := \begin{pmatrix} \alpha_{1,x}\beta_2 - \alpha_2\beta_{1,x} \\ \alpha_{2,x}\beta_2 - \alpha_2\beta_{2,x} \\ \int (\alpha_1\beta_{1,x} - \alpha_{1,x}\beta_1)\, \mathrm{d}x \end{pmatrix}. \tag{64}$$

Obviously $\mathcal{G}$ is a central extension of its subalgebra determined by the constraint $\alpha_3(x) \equiv 0$. The dual space $\mathcal{G}^*$ of $\mathcal{G}$ is given by the set of elements of the form

$$\boldsymbol{u}(z, x) = \begin{pmatrix} v(z, x) \\ u(z, x) \\ c(x) \end{pmatrix}, \qquad v(\cdot, x), u(\cdot, x) \in \mathbb{C}((z)),$$

acting on $\mathcal{G}$ according to

$$\langle \boldsymbol{u}, \boldsymbol{\alpha} \rangle := \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} \left(c\alpha_3 + \int \mathrm{d}x (v\alpha_1 + u\alpha_2) \right). \tag{65}$$

It is straightforward to find that the coadjoint action of $\mathcal{G}$ on $\mathcal{G}^*$

$$\langle ad^*\boldsymbol{\alpha}(\boldsymbol{u}), \boldsymbol{\beta} \rangle = \langle \boldsymbol{u}, [\boldsymbol{\beta}, \boldsymbol{\alpha}] \rangle,$$

is given by

$$ad^*\boldsymbol{\alpha}(\boldsymbol{u}) = \begin{pmatrix} -2\alpha_{1,x} + (v\alpha_2)_x \\ v\alpha_{1,x} + 2u\alpha_{2,x} + u_x\alpha_2 \\ 0 \end{pmatrix} = \begin{pmatrix} J(v, u) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \\ 0 \end{pmatrix}. \tag{66}$$

Hence we conclude that the operator $J(v, u)$ of (61) is the symplectic operator corresponding to the Lie–Poisson bracket of $\mathcal{G}$.

The functionals (see (63))

$$C_N[\boldsymbol{u}] := \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} \int \mathrm{d}x\, \frac{z^N}{2} \sqrt{v^2 + 4u},$$

represent Casimir invariants as they satisfy

$$J(v, u)\nabla C_N = 0. \tag{67}$$

Note that in order to ensure that $\nabla C_N(\boldsymbol{u}) \in \mathcal{G}$, we must restrict C_N to the subset of elements of $\mathcal{G}^*$ for which degree$(v^2 + 2u)$ at $z \to \infty$ is even.

On the other hand, there is an infinite family of R-matrices in $\mathcal{G}$ of the form

$$R_m\boldsymbol{\alpha} := \tfrac{1}{2}((z^m\boldsymbol{\alpha})_+ - (z^m\boldsymbol{\alpha})_-), \qquad m \in \mathbb{Z}. \tag{68}$$

They define a corresponding family of Lie-algebra structures in $\mathcal{G}$ given by

$$[\boldsymbol{\alpha}, \boldsymbol{\psi}]_m = [R_m\boldsymbol{\alpha}, \boldsymbol{\psi}] + [\boldsymbol{\alpha}, R_m\boldsymbol{\psi}]. \tag{69}$$

According to the general theory [33], the Casimir invariants of $(\mathcal{G}, [\,,\,])$ are in involution

$$\{C_N, C_M\}_m = 0, \qquad N, M \geqslant 0,$$

with respect to all the Poisson–Lie bracket structures $(\mathcal{G}, [\,,\,]_m)$, $m \in \mathbb{Z}$.

Therefore for each $m \in \mathbb{Z}$, there is an associated family of compatible Hamiltonian systems

$$\begin{aligned}\partial_{t_{m,N}}\boldsymbol{u} &= \tfrac{1}{2}(ad^*(R_m\nabla C_N(\boldsymbol{u})))(\boldsymbol{u}) = (ad^*(z^m\nabla C_N(\boldsymbol{u}))_+)(\boldsymbol{u}) \\ &= J(v,u)(z^m\nabla C_N(\boldsymbol{u}))_+ = -J(v,u)(z^m\nabla C_N(\boldsymbol{u}))_-.\end{aligned} \tag{70}$$

These Hamiltonian flows are defined on the invariant submanifold $\mathcal{D}$ given by the elements $\boldsymbol{u}$ of $\mathcal{G}^*$ such that

$$d_1 \leqslant d_0 + 1, \qquad d = \max(2d_1, d_0) = \text{even}, \tag{71}$$

with d_0 and d_1 being the degrees of u and v as $z \to \infty$, respectively. Moreover, given positive integers d_0 and d_1 verifying (71) then the submanifols $\mathcal{D}_{d_0,d_1}$ of $\mathcal{D}$ with elements of the form

$$v = \sum_{n=0}^{d_1} v_n z^n, \qquad u = \sum_{n=0}^{d_0} u_n z^n, \tag{72}$$

are left invariant under the flows (70).

Therefore we conclude that under conditions (71), the flows (61) form a hierarchy of compatible Hamiltonian systems for (v, u).

Furthermore, as a consequence of the identities

$$\nabla C_{N+m}(\boldsymbol{u}) = z^m\nabla C_N,$$

it follows that the flows (70) have an infinite number of compatible Hamiltonian formulations. On the other hand, one finds that the coadjoint action of $(\mathcal{G}, [\,,\,]_m)$ on $\mathcal{G}^*$ reads

$$2ad^*_m\alpha(\boldsymbol{u}) = J(v,u)(R_m\alpha) - z^m R(J(v,u)\alpha).$$

In particular, (71) implies

$$\begin{aligned}ad^*_m\alpha(\boldsymbol{u}) &= J(v,u)(z^m\alpha_-)_+ - z^m(J(v,u)\alpha_-)_+ \\ &= z^m(J(v,u)\alpha_-)_- - J(v,u)(z^m\alpha_-)_-,\end{aligned}$$

and it means that the images of v and u under $ad^*_m\alpha$ are polynomials in z with degrees

$$\tilde{d}_1 = \max(m-1, d_1-1), \qquad \tilde{d}_2 = \max(m-1, d_1-1, d_0-1).$$

Therefore, for $0 \leqslant m \leqslant d_1$ the subsets (72) determine Poisson submanifolds of $\mathcal{G}^*$. In this way, we conclude that under conditions (71), equations (61) not only form a hierarchy of compatible Hamiltonian systems for (v, u) but also have $d_1 + 1$ different Hamiltonian structures.

The same result holds for (61) if $d = \max(2d_1, d_0)$ is an odd integer. To prove it one applies the above analysis to the submanifols $\widetilde{\mathcal{D}}_{d_0,d_1}$ of $\mathcal{D}$ with elements of the form

$$v = \sum_{n=0}^{d_1} v_n z^{2n}, \qquad u = \sum_{n=0}^{d_0} u_n z^{2n}. \tag{73}$$

There are many interesting reductions of (61). For example a compatible constraint is $v \equiv 0$. That is to say

$$F := p^2 - u(k) = 0,$$

so that $p_+ = -p_-$ and (61) becomes

$$\partial_t u = J\left(\frac{\delta C_N}{\delta u}\right)_+,$$

where

$$J := J(u) = 2u\partial_x + u_x,$$
$$C_N[u] := \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\int \mathrm{d}x\, z^N p_+ = \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}}\int \mathrm{d}x\, z^N\sqrt{u}.$$

It can be seen that for this case our analysis leads to the Hamiltonian description obtained from the geometrical approach in [34].

4.2. Riemann invariants and gauge transformations

The discriminant of $F = p^2 - vp - u$ is given by

$$R(z) = -(v^2 + 4u),$$

so that we can formulate the flows (61) in terms of the gauge invariant variable $w := v^2 + 4u$ and the gauge variable $\rho := v/2$. Thus, from (62) and (63), one finds

$$\partial_{t_N} w = (2w\partial_x + w_x)\left(\frac{z^N}{\sqrt{w}}\right)_+, \qquad \partial_{t_N}\rho = -\partial_x\left(\left(\frac{z^N}{\sqrt{w}}\right)_-\rho\right)_+. \tag{74}$$

Observe that the variable w evolves independently of ρ and that the equation for ρ_t is linear in ρ. This suggests the following scheme for the integration of (61): we integrate first the equation for w in terms of Riemann invariants and then we solve the linear equation for ρ.

Example. The t_1-flow for

$$F := p^2 - (z + v_0)p - (z^2 + zu_1 + u_0) = 0,$$

is

$$\begin{aligned} &\partial_t v_0 = \tfrac{1}{5\sqrt{5}}(v_{0,x} + 2u_{1,x}), \qquad \partial_t u_1 = \tfrac{1}{5\sqrt{5}}(2v_{0,x} + 4u_{1,x}), \\ &\partial_t u_0 = \tfrac{1}{5\sqrt{5}}(2v_0 v_{0,x} - v_0 u_{1,x} + u_{0,x}). \end{aligned} \tag{75}$$

In this case, we have the gauge invariants

$$w := v^2 + 4u = 5z^2 + 2zw_1 + w_2, \qquad w_1 = v_0 + 2u_1, \qquad w_2 = v_0^2 + 4u_0.$$

They are also Riemann invariants as they verify

$$\partial_t w_i = \tfrac{1}{\sqrt{5}}\partial_x w_i, \qquad i = 1, 2.$$

Moreover, the gauge variable $\rho_0 := v_0/2$ evolves according to

$$\partial_t \rho_0 = \tfrac{1}{10\sqrt{5}}\partial_x w_1.$$

In this way, the integration of (75) reduces to elementary operations.

4.3. Dispersionful counterpart

There is a dispersionful version of the hierarchy of flows (61) which can be described in terms of the energy-dependent spectral problem

$$L\psi := \partial_{xx}\psi - v(k,x)\psi_x - u(k,x)\psi = 0, \tag{76}$$

where

$$v = \sum_{i=0}^{d_1} v_i k^i, \qquad u = \sum_{i=0}^{d_0} u_i k^i.$$

Indeed,the compatibility between (76) and flows of the form

$$\partial_t \psi = a\psi + b\psi_x, \qquad a, b \in \mathbb{C}[z],$$

where we are assuming (58), leads us to the following equations for v and u

$$\partial_t \begin{pmatrix} v \\ u \end{pmatrix} = \begin{pmatrix} 2\partial_x & \partial_{xx} + \partial_x(v\cdot) \\ \partial_{xx} - v\partial_x & 2u_0\partial_x + u_{0,x} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{77}$$

Thus we consider a Lenard relation

$$\begin{pmatrix} -2\partial_x & \partial_{xx} + \partial_x(v\cdot) \\ -\partial_{xx} + v\partial_x & 2u_0\partial_x + u_{0,x} \end{pmatrix} \begin{pmatrix} R \\ S \end{pmatrix} = 0,$$

which reduces to

$$R = \tfrac{1}{2}(S_x + vS), \qquad -\tfrac{1}{2}S_{xxx} + 2US_x + U_x S = 0, \tag{78}$$

where

$$U := -\tfrac{1}{2}v_x + \tfrac{1}{4}v^2 + u.$$

The first equation of (78) is satisfied if

$$\begin{pmatrix} R \\ S \end{pmatrix} = \begin{pmatrix} \frac{\delta C}{\delta v} \\ \frac{\delta C}{\delta u} \end{pmatrix},$$

with

$$C := C[U] = \oint_\gamma \frac{\mathrm{d}z}{2\pi\mathrm{i}} \int \mathrm{d}x\, \chi(U),$$

being a functional depending on U. As for the second equation of (78), it reduces to

$$\chi_x + \chi^2 - U = 0,$$

which in turn is satisfied by

$$\chi = \sigma - \tfrac{1}{2}v,$$

with σ being a solution of

$$\sigma_x + \sigma^2 - v\sigma - u = 0.$$

The last equation is verified by

$$\sigma := \partial_x \ln \psi.$$

Therefore, we have found a quantum counterpart of the hierarchy (61)

$$\partial_{t_N} \boldsymbol{u} = \mathcal{J}(\nabla H_N)_+, \qquad N \geqslant 0, \tag{79}$$

where

$$\mathcal{J} := \begin{pmatrix} -2\partial_x & \partial_{xx} + \partial_x(v\cdot) \\ -\partial_{xx} + v\partial_x & 2u\partial_x + u_x \end{pmatrix} \tag{80}$$

and

$$H_N[v, u] := \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} \int \mathrm{d}x\, z^N \chi(U).$$

We note that

$$H_N[v, u] = \oint_\gamma \frac{\mathrm{d}z}{2\pi \mathrm{i}} \int \mathrm{d}x\, \frac{z^N}{2}(\sigma_+ - \sigma_-),$$

where

$$\sigma_\pm = \tfrac{1}{2}v \pm \chi,$$

verify

$$L\psi = (\partial_x - \sigma_-)(\partial_x - \sigma_+).$$

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